Group: Probability and statistics

Problem 1. Let $p \in (0,1)$. Suppose you have a coin with probability of "head" unknown. Can you design a game between two persons, which ends in a finite number of tosses of the coin with probability 1, such that one person's winning probability is exactly p.

Problem 2. Let (p_1, p_2, p_3) be the probabilities of a discrete random variable, where $0 < p_k < 1$ and $p_1 + p_2 + p_3 = 1$. Let r_1, r_2, r_3 be independent random variables, each following a Unif$(0,1)$ distribution. Define the random variable X as

$$X = k, \quad \text{if } r_k^{1/p_k} = \max\left\{r_1^{1/p_1}, r_2^{1/p_2}, r_3^{1/p_3}\right\}.$$

You may ignore the case where the maximum is not unique, as it occurs with probability zero. Determine the distribution of X.

Problem 3. Let $(X_n)_{n\geq 0}$ be a discrete time simple symmetric random walk on $\mathbb{Z}^d$, whose increments $(X_{n+1} - X_n)_{n\geq 0}$ are independent and chosen uniformly from the $2d$ unit vectors $(\pm e_i)_{1\leq i\leq d}$ in $\mathbb{Z}^d$ with $\|e_i\| = 1$. A function $f : \mathbb{Z}^d \to \mathbb{R}$ is called *harmonic* for the random walk if $\mathbb{E}[f(X_1)|X_0 = x] = f(x)$ for all $x \in \mathbb{Z}^d$. Show that every *bounded harmonic function* f is a constant.